

# The Froude number for solitary water waves with vorticity

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## Abstract

We consider two-dimensional solitary water waves on a shear flow with an arbitrary distribution of vorticity. Assuming that the horizontal velocity in the fluid never exceeds the wave speed and that the free surface lies everywhere above its asymptotic level, we give a very simple proof that a suitably defined Froude number  $F$  must be strictly greater than the critical value  $F = 1$ . We also prove a related upper bound on  $F$ , and hence on the amplitude, under more restrictive assumptions on the vorticity.

## 1 Introduction

### 1.1 Statement of the main results

We consider the motion of a two-dimensional fluid which is bounded above by a free surface under constant (atmospheric) pressure and below by a horizontal bed. Gravity acts as an external force, and there is no surface tension on the free surface. Inside the fluid, the velocity  $(u, v)$  and pressure  $P$  satisfy the incompressible Euler equations. We denote the horizontal bed by  $y = -d$  and the free surface by  $y = \eta(x, t)$ . Fixing the constant wave speed  $c > 0$ , we assume that the motion is steady in that  $\eta$ ,  $u$ ,  $v$ , and  $P$  depend on  $x$  and  $t$  only through the combination  $x - ct$ , which we henceforth abbreviate to  $x$ . We also assume that the wave is solitary in that

$$\eta \rightarrow 0, \quad v \rightarrow 0, \quad u \rightarrow U(y), \quad \text{as } x \rightarrow \pm\infty,$$

uniformly in  $y$ , where the horizontal velocity  $U$  of the shear flow at  $x = \pm\infty$  is an arbitrary function of  $-d \leq y \leq 0$ . We call a solitary wave *trivial* if  $\eta \equiv 0$ ,  $v \equiv 0$ , and  $u \equiv U$ . In the context of this paper, we will also call a solitary wave a *wave of elevation* if  $\eta(x) \geq 0$  for all  $x$  but  $\eta \not\equiv 0$ . Similarly we call a solitary wave a *wave of depression* if  $\eta(x) \leq 0$  for all  $x$  but  $\eta \not\equiv 0$ .

The classical Froude number for solitary waves is the dimensionless ratio  $c/\sqrt{gd}$ . When working with shear flows, however, we find it more convenient to define the Froude number  $F$  by

$$\frac{1}{F^2} = g \int_{-d}^0 \frac{dy}{(c - U(y))^2}, \quad (1.1)$$

where we have assumed that the velocity  $U$  of the shear flow is strictly less than the wave speed  $c$ . This definition reduces to the classical one when  $U$  vanishes identically, and it has the advantage that the critical Froude number is  $F = 1$  regardless of the shear flow  $U$ . In particular, with this convention the small-amplitude solitary waves with vorticity constructed in [TK61, Hur08a, GW08] have Froude numbers  $F$  slightly bigger than 1.

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The reader may assume that  $u, v, P, \eta, U$  are all  $C^2$  or even smooth. On the other hand, our arguments go through unchanged for solutions with the more limited regularity

$$u, v, P \in W_{\text{loc}}^{1,r}(\overline{D_\eta}) \subset C_{\text{loc}}^\alpha(\overline{D_\eta}), \quad \eta \in C^{1+\alpha}(\mathbb{R}), \quad U \in W^{1,r}(-d, 0) \subset C^\alpha[-d, 0], \quad (1.2)$$

where here  $0 < \alpha < 1$ ,  $r = 2/(1 - \alpha)$ , and  $D_\eta = \{(x, y) : -d < y < \eta(x)\}$  denotes the fluid domain. By  $w \in W_{\text{loc}}^{1,r}(\overline{D_\eta})$  we mean that  $w \in W^{1,r}(D')$  whenever  $\overline{D'} \subset \overline{D_\eta}$  is compact, and similarly for  $C_{\text{loc}}^\alpha(\overline{D_\eta})$ . The regularity (1.2) is an analogue for solitary waves of the regularity assumed in Theorem 2 of [CS11] for periodic waves.

**Theorem 1.1.** *Consider a solitary wave with  $\sup u < c$  and the regularity (1.2). Then  $F \neq 1$ . Moreover,  $F > 1$  if it is a wave of elevation, and  $F < 1$  if it is a wave of depression.*

In the irrotational case where the vorticity  $\omega = v_x - u_y$  vanishes identically and  $U$  is constant, the assumption  $\sup u < c$  is automatically satisfied [Tol96] and the bound  $F > 1$  for waves of elevation is well-known [Sta47, AT81, McL84]. While the assumption  $\sup u < c$  is still reasonable for waves with vorticity [CS04], it rules out the existence of critical layers or stagnation points in flow.

In some cases, the argument leading to Theorem 1.1 can be extended to give an upper bound on the Froude number for waves of elevation. Before giving this result, we define a dimensionless quantity  $\Lambda \geq 1$  by

$$\Lambda = \max_y \frac{c - U(0)}{c - U(y)}. \quad (1.3)$$

We emphasize that  $\Lambda$ , like  $F$ , only depends on the shear flow  $U$  at infinity.

**Theorem 1.2.** *For any solitary wave of elevation with the regularity (1.2),  $\sup u < c$ , and  $\Lambda < 2/\sqrt{3}$ , the Froude number  $F$  satisfies the upper bound*

$$F < (1 - \frac{3}{4}\Lambda^2)^{-1/2}. \quad (1.4)$$

For irrotational waves,  $U$  is constant, so clearly  $\Lambda = 1 < 2/\sqrt{3}$  and hence Theorem 1.2 gives the well-known bound  $F < 2$  [Sta47, AT81, McL84]. More generally, if the vorticity  $\omega \leq 0$ , then  $U(y) \leq U(0)$  for  $-d \leq y \leq 0$  so that again  $\Lambda = 1$ , and Theorem 1.2 gives the same bound  $F < 2$ . In terms of the antiderivative  $\Gamma(p)$  of the vorticity function and Bernoulli constant  $\lambda$  defined in Section 2, the condition  $\Lambda < 2/\sqrt{3}$  can be rephrased as  $\min_p \Gamma(p) > -\lambda/8$ .

## 1.2 Historical discussion

**Irrotational waves.** For irrotational waves, the asymptotic shear flow  $U$  is constant, and can be taken to be zero by switching to an appropriate reference frame. Our formula (1.1) for  $F$  then reduces to the classical ratio

$$F = \frac{c}{\sqrt{gd}},$$

which is named in honor of William Froude, who in the 1870s argued that in order to compare the resistances felt by scaled models of a ship, the ratio of the speed of the ship to the square root of its length must be kept constant [Fro74].

The importance of the critical speed  $c = \sqrt{gd}$  corresponding to  $F = 1$  was known long before Froude. In 1781, Lagrange showed that long irrotational waves in shallow water travel with nearly

this speed [Dar03], and in 1828 Bélanger showed that a hydraulic jump can occur only if the upstream flow has  $F > 1$  [Bél28, Cha09]. More pertinent to this article is John Scott Russell’s famous 1844 report [Rus44], which gives the empirical formula

$$F^2 \approx 1 + \frac{\max \eta}{d} \quad (1.5)$$

for the speed of small-amplitude irrotational solitary waves. Theoretical justifications of (1.5) came decades later with the work of Boussinesq in 1871 and Rayleigh in 1876 [Dar03].

In 1947, Starr gave a formal proof [Sta47] of the strikingly simple exact formula

$$F^2 = 1 + \frac{3}{2d} \frac{\int \eta^2 dx}{\int \eta dx} \quad (1.6)$$

for irrotational solitary waves; see [LH74] for an alternate derivation. Given this identity, the upper and lower bounds  $1 < F < 2$  for waves of elevation are straightforward. Indeed, since  $\eta \geq 0$  does not vanish identically, (1.6) immediately implies  $F > 1$ . On the other hand, (1.6) also implies the upper bound

$$F^2 < 1 + \frac{3}{2} \frac{\max \eta}{d}. \quad (1.7)$$

Since  $\max \eta \leq F^2 d/2$  by Bernoulli’s law, (1.7) in turn implies  $F^2 < 1 + 3F^2/4$  and hence  $F < 2$ . Substituting  $F < 2$  back into Bernoulli’s law we also obtain the bound  $\max \eta < 2d$  on the amplitude.

In fact, Starr showed the improved upper bound

$$F^2 < 1 + \frac{\max \eta}{d}. \quad (1.8)$$

in which the coefficient  $3/2$  in (1.7) has been reduced to the  $1$  appearing in the asymptotic formula (1.5). See [KP74] for an alternate derivation. Arguing as in the previous paragraph, (1.8) leads to the bounds  $F < \sqrt{2}$  and  $\max \eta < d$  on the Froude number and amplitude. We note that the proofs of (1.8) in [Sta47, KP74] do not depend on the identity (1.6).

Amick and Toland gave rigorous proofs of the bounds  $1 < F < 2$  in their construction of large-amplitude irrotational solitary waves [AT81], in which the water wave problem is reformulated as a Nekrasov-type integral equation on the free surface. They objected to the assumption that the “mass”  $\int \eta dx$  was finite in the earlier proofs [Sta47, LH74, KP74], and instead, as McLeod puts it [McL84], “take sixteen pages and much complicated estimating of integrals to prove  $F > 1$  without the assumption of finite mass”. In response to [AT81], McLeod [McL84] showed that the earlier proofs could be easily modified to avoid the assumption of finite mass. This modified proof also shows that the mass is necessarily finite.

For later reference we also mention a third upper bound

$$F^2 < \frac{2(1 + \max \eta/d)^2}{2 + \max \eta/d}, \quad (1.9)$$

which was obtained by Keady and Pritchard [KP74] using maximum principle arguments. It is easy to check that (1.9) is strictly weaker than (1.6) and (1.8). In particular, it cannot be combined with Bernoulli’s law to obtain a bound on the Froude number which is independent of the amplitude  $\max \eta$ .

Numerics suggest that there is a one-parameter family of irrotational solitary waves connecting small-amplitude waves with  $F$  slightly bigger than  $1$  and the so-called wave of greatest height,

which has a stagnation point at its crest where there is a corner with a  $120^\circ$  interior angle. The maximum value of the Froude number as well as the maxima of mass, momentum, and energy for this family are all achieved before the wave of greatest height is reached [Mil80, LHF74]. The wave with maximum Froude number has  $F = 1.294$  and  $\max \eta/d = 0.790$ . There seems to be some disagreement about the precise value of the maximum amplitude, but nevertheless a consensus that it is approximately  $\max \eta/d = 0.83$  and hence that the corresponding Froude number is  $F = 1.29$  [Mil80, LHF74, HVB83]. We note that the existence of a wave of extreme form was proved rigorously in [AT81, AFT82], while the existence of bifurcation or turning points in the connected set of solutions containing small-amplitude solitary waves was proved in [Plo91].

**Waves with vorticity.** With vorticity, the importance of the critical value  $F = 1$  for long waves in shallow water was recognized by Burns in 1953 [Bur53];  $F = 1$  is sometimes called the “Burns condition”. The small-amplitude solitary waves constructed by Ter-Krikorov [TK61] in 1961 and later by Hur [Hur08a] and then Groves and Wahlén [GW08] all have  $F$  slightly bigger than 1.

Besides the existence results mentioned above, there are to our knowledge no lower bounds in the literature on the Froude number for solitary waves with vorticity. Moreover, the only upper bounds [KK12, Whe13] (also see [KN82]) are proved by maximum principle arguments and reduce to (1.9) for irrotational waves. In particular, these bounds seem not to lead to bounds on the Froude number which are independent of the amplitude  $\max \eta$ .

There are, however, many results on solitary waves with vorticity which require the assumption that  $F > 1$ . In particular, Hur proved exponential asymptotics [Hur08b] and analyticity of streamlines [Hur11] for waves with  $F > 1$ , and symmetry for waves with  $F > 1$  which are also waves of elevation [Hur08b] (see [MM12] for related results without the assumption  $F > 1$ ). Upper and lower bounds (or the lack thereof) on the Froude number also feature prominently in the construction by the author of large-amplitude solitary waves in [Whe13], where it was also shown that waves with  $F > 1$  are necessarily strict waves of elevation in that  $\eta(x) > 0$  for all  $x$ . See Section 5 for more on the consequences of our main results for the amplitude, elevation, symmetry, monotonicity, and decay of solitary waves, and Section 6 for more on the existence of large-amplitude waves.

With large positive constant vorticity, numerics seem to suggest the existence of overhanging waves with arbitrarily large Froude number [VB94], a phenomena which cannot occur for irrotational waves. See Section 7 for versions of Theorems 1.1 and 1.2 in the special case of constant vorticity. Note that overhanging waves must have  $u = c$  somewhere along their free surfaces, and hence do not satisfy the hypothesis  $\sup u < c$  of Theorems 1.1 and 1.2. While the existence proofs in [TK61, Hur08a, GW08] are restricted to waves with  $\sup u < c$ , the qualitative results in [KK12] are not.

### 1.3 Method of proof and further consequences

**Integral identities with vorticity.** The main ingredient in the proof of Theorem 1.1 is an integral identity, Lemma 3.1, which seems to be completely new. In particular, we emphasize that this lemma is *not* a straightforward generalization of the identity (1.6) used in the irrotational case. Indeed, following the proof of (1.6) but retaining the extra terms coming from vorticity, one obtains a different identity, Lemma 4.2 (which we use in the proof of Theorem 1.2). Under the relatively strong assumption that vorticity is nonnegative, Lemma 4.2 implies a lower bound on the relative speed  $c - U(0)$  of asymptotic shear flow at the free surface. Except in the irrotational case where  $U(y)$  is constant, however, this bound is not sufficient to show that the Froude number

$F$  is greater than the critical value 1. Indeed, the definition (1.1) of  $F$  involves the values  $U(y)$  for all  $-d \leq y \leq 0$ , not just  $y = 0$ , and the lower bound  $F > 1$  is sharp for small-amplitude waves.

**Existence of large-amplitude waves.** In [Whe13], the author constructed a connected set  $\mathcal{C}$  of solitary waves of elevation whose asymptotic shear flows are given by

$$U(y) = U(y; F) = c - FU^*(y) \quad (1.10)$$

for some arbitrary but fixed positive function  $U^*$  satisfying a normalization condition. It is easy to see from (1.10) that the dimensionless parameter  $\Lambda$  is constant along  $\mathcal{C}$ ; when  $\Lambda < 2/\sqrt{3}$ , we can combine Theorems 1.1 and 1.2 with the results in [Whe13] to show that there exists a sequence of waves in  $\mathcal{C}$  which approach stagnation in that  $\sup u_n \rightarrow c$ . This significant improvement is presented in detail in Section 6 (Corollary 6.2), along with a related result for  $\Lambda \geq 2/\sqrt{3}$  (Corollary 6.1).

**Waves with surface tension.** For irrotational waves with surface tension, Amick and Kirchgässner [AK89] prove the analogue

$$F^2 = 1 + \frac{3}{2d} \frac{\int \eta^2 dx}{\int \eta dx} + \frac{\sigma}{gd} \frac{\int (\sqrt{1 + \eta_x^2} - 1) dx}{\int \eta dx} \quad (1.11)$$

of (1.6). When the constant coefficient  $\sigma$  of surface tension is nonnegative, (1.11) implies that solitary waves of elevation have  $F > 1$  while solitary waves of depression have  $F < 1$ . In Section 8, we will show that the same is true for waves with vorticity; indeed, with slightly stronger regularity assumptions, Lemma 3.1 and Theorem 1.1 continue to hold in the presence of surface tension and with nearly identical proofs.

## 1.4 Outline

In Section 2, we perform a standard change of variables and collect some formulas which will be used later. The main benefit of this change of variables is that it transforms the fluid domain into a fixed infinite strip, enabling us to multiply the Euler equations by functions defined in terms of the asymptotic shear flow  $U(y)$  appearing in the definition (1.1) of the Froude number  $F$ .

In Section 3, we give a short and elementary proof of Theorem 1.1, based on an integral identity, Lemma 3.1, in the transformed variables. The argument is perhaps even simpler than the argument in [McL84], provided the change of variables in Section 2 is taken for granted.

In Section 4, we prove the upper bound Theorem 1.2 using two additional integral identities. The first, Lemma 4.2, is essentially (1.6) with an extra term coming from the vorticity, and is proved mostly in the original physical variables. The second, Lemma 4.3 is proved similarly to Lemma 3.1, but with a different test function.

In Section 5, we collect some implications of Theorems 1.1 and 1.2 for the amplitude, mass, symmetry, monotonicity, and exponential decay of solitary waves of elevation.

In Section 6, we show how Theorems 1.1 and 1.2 can be used to improve the existence theory for large-amplitude solitary waves with vorticity developed by the author in [Whe13].

In Section 7, we specialize Theorems 1.1 and 1.2 to the case where the vorticity is constant and more explicit formulas can be given.

Finally, in Section 8 we prove that, with slightly modified regularity assumptions, Lemma 3.1 and Theorem 1.1 still hold for waves with surface tension.

## 2 Preliminaries

With the conventions from Section 1.1, we assume that  $u, v, P \in W_{\text{loc}}^{1,r}(\overline{D_\eta})$  satisfy the stationary incompressible Euler equations

$$(u - c)u_x + vu_y = -P_x, \quad (2.1a)$$

$$(u - c)v_x + vv_y = -P_y - g, \quad (2.1b)$$

$$u_x + v_y = 0, \quad (2.1c)$$

in  $L_{\text{loc}}^r(\overline{D_\eta})$ , together with the boundary conditions

$$v = 0 \quad \text{on } y = -d, \quad (2.1d)$$

$$v = (u - c)\eta_x \quad \text{on } y = \eta(x), \quad (2.1e)$$

$$P = 0 \quad \text{on } y = \eta(x), \quad (2.1f)$$

pointwise, and the asymptotic conditions

$$\eta \rightarrow 0, \quad v \rightarrow 0, \quad u \rightarrow U(y) \quad \text{as } x \rightarrow \pm\infty, \quad (2.1g)$$

uniformly in  $y$ . The free surface elevation  $\eta$  has the regularity  $\eta \in C^{1+\alpha}(\mathbb{R})$ . Here for convenience we have normalized  $P$  to vanish on the free surface;  $P$  therefore represents the difference between the pressure in the fluid and the atmospheric pressure.

Because of incompressibility (2.1c), there exists a stream function  $\psi \in W_{\text{loc}}^{2,r}(\overline{D_\eta}) \subset C_{\text{loc}}^{1+\alpha}(\overline{D_\eta})$  which satisfies  $\psi_y = u - c$  and  $\psi_x = v$ . From now on we will always assume  $\sup u < c$ , or equivalently  $\sup \psi_y < 0$ . By the kinematic boundary conditions (2.1d)–(2.1e),  $\psi$  is constant on  $y = -d$  and  $y = \eta(x)$ . Thus the flux

$$m := \psi(x, -d) - \psi(x, \eta(x)) = \int_{-d}^{\eta(x)} (c - u(x, y)) dy = \int_{-d}^0 (c - U(y)) dy \quad (2.2)$$

is independent of  $x$ . We normalize  $\psi$  so that  $\psi = 0$  on  $y = \eta(x)$  and  $\psi = -m$  on  $y = -d$ . The vorticity  $\omega$  is given in terms of  $\psi$  by

$$\omega = v_x - u_y = -\Delta\psi = \gamma(\psi)$$

for some function  $\gamma \in L^r[-m, 0]$  called the *vorticity function* [CS11].

Using

$$q = x, \quad p = -\psi$$

as independent variables, we can rewrite (2.1) in terms of the so-called height function  $h(q, p)$  defined by

$$h(x, -\psi(x, y)) = y + d. \quad (2.3)$$

The advantage of this formulation is that  $h$  is defined on the fixed domain

$$\Omega := \{(q, p) : -m < p < 0\}.$$

Defining the asymptotic height function  $H(p)$  in a similar way,

$$H(-\Psi(y)) = y + d, \quad \text{where } \Psi(y) = \int_0^y (U(s) - c) ds, \quad (2.4)$$

we have  $H(0) = d$  and  $H(-m) = 0$ . Since (2.3) implies  $h_p^{-1} = -\psi_y$ , we necessarily have  $\inf h_p > 0$  and  $\min H_p > 0$ .

The arguments in [CS11] show that, under the crucial assumption  $\sup u < c$ , the solitary water wave problem (2.1) is equivalent to the system

$$\left( -\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right)_p + \left( \frac{h_q}{h_p} \right)_q = 0 \quad -m < p < 0, \quad (2.5a)$$

$$\frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} + g(h-H) = 0 \quad \text{on } p = 0, \quad (2.5b)$$

$$h = 0 \quad \text{on } p = -m, \quad (2.5c)$$

for  $h \in W_{\text{loc}}^{2,r}(\bar{\Omega}) \subset C_{\text{loc}}^{1+\alpha}(\bar{\Omega})$  and  $H \in W^{2,r}[-m, 0]$  together with the asymptotic conditions

$$h_p \rightarrow H_p, \quad h_q \rightarrow 0 \text{ as } q \rightarrow \pm\infty \quad \text{uniformly in } p. \quad (2.5d)$$

The velocity field  $(u, v)$  and free surface  $\eta$  can be recovered from  $h$  via

$$c - u = \frac{1}{h_p}, \quad v = -\frac{h_q}{h_p}, \quad \eta(q) = h(q, 0) - H(0). \quad (2.6)$$

We note that the divergence-form equation (2.5a) expresses the balance of the  $y$ -component of momentum and that (2.5b) is Bernoulli's law evaluated restricted to the free surface.

Defining the antiderivative  $\Gamma \in W^{1,r}[-m, 0] \subset C^\alpha[-m, 0]$  of the vorticity function  $\gamma$  and the Bernoulli constant  $\lambda$  by

$$\Gamma(p) = \int_0^p \gamma(-s) ds, \quad \lambda = (U(0) - c)^2 = \frac{1}{H_p^2(0)}, \quad (2.7)$$

we have the following useful relation between  $\gamma$ ,  $H$ , and  $U$

$$(U - c)^2(H(p)) = \frac{1}{H_p^2(p)} = \lambda + 2\Gamma(p). \quad (2.8)$$

In particular, the Froude number  $F$  is given in terms of  $H$  by

$$\frac{1}{F^2} = g \int_{-d}^0 \frac{dy}{(U(y) - c)^2} = g \int_{-m}^0 H_p^3(p) dp. \quad (2.9)$$

In addition, Bernoulli's law can be written as

$$P + \frac{(u - c)^2 + v^2}{2} + gy - \frac{\lambda}{2} - \Gamma(-\psi) \equiv 0. \quad (2.10)$$

Indeed, one can see that the left hand side is constant by differentiating and using (2.1a)–(2.1b). The fact that this constant is zero follows by sending  $x \rightarrow \pm\infty$  in the the dynamic boundary condition (2.1f).

### 3 Lower bound

**Lemma 3.1.** *Any solitary wave with  $\sup u < c$  and the regularity (1.2) satisfies*

$$\left( \frac{1}{F^2} - 1 \right) \int_{-M}^M \eta dx + \int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2} dp dq \rightarrow 0 \quad \text{as } M \rightarrow \infty. \quad (3.1)$$



*Proof.* Defining the function

$$\Phi(p) = \int_{-m}^p H_p^3(s) ds, \quad (3.2)$$

we note that (2.9) implies  $g\Phi(0) = 1/F^2$ . Multiplying (2.5a) by  $\Phi$  and integrating by parts, we then have, for any  $M > 0$ ,

$$\begin{aligned} 0 &= \int_{-M}^M \int_{-m}^0 \left[ \left( -\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right)_p \Phi + \left( \frac{h_q}{h_p} \right)_q \Phi \right] dp dq \\ &= \int_{-M}^M \int_{-m}^0 \left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p^3 dp dq + \frac{1}{gF^2} \int_{-M}^M \left( -\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right) (q, 0) dq \\ &\quad + \int_{-m}^0 \frac{h_q}{h_p} \Phi dp \Big|_{x=-M}^{x=M}. \end{aligned} \quad (3.3)$$

Since  $h_q \rightarrow 0$  as  $q \rightarrow \pm\infty$  by (2.5d), the third term in (3.3) vanishes as  $M \rightarrow \infty$ . Using the boundary condition (2.5b) to simplify the second term, we obtain

$$\int_{-M}^M \int_{-m}^0 \left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p^3 dp dq + \frac{1}{F^2} \int_{-M}^M (h(q, 0) - H(0)) dq \rightarrow 0 \quad (3.4)$$

as  $M \rightarrow \infty$ . Rewriting the first integrand in (3.4) as

$$\left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p^3 = -(h_p - H_p) + \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2}, \quad (3.5)$$

we see that

$$\begin{aligned} &\int_{-M}^M \int_{-m}^0 \left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p^3 dp dq \\ &= - \int_{-M}^M (h(q, 0) - H(0)) dq + \int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2} dp dq. \end{aligned} \quad (3.6)$$

Plugging (3.6) into (3.4), rearranging terms, and using the identity  $h(q, 0) - H(0) = \eta(q)$  from (2.6), we obtain (3.1) as desired.  $\square$

*Proof of Theorem 1.1.* Consider a nontrivial solitary wave. Then  $H_p$  and  $h_p$  are strictly positive, and  $h_q(q, 0) = \eta_x(q)$  does not vanish identically. Thus the second integrand in (3.1) is a nondecreasing function of  $M$  and is strictly positive for  $M$  sufficiently large, and therefore the limit in (3.1) implies

$$\limsup_{M \rightarrow \infty} \left\{ \left( \frac{1}{F^2} - 1 \right) \int_{-M}^M \eta dx \right\} < 0. \quad (3.7)$$

Since the left hand side of (3.7) vanishes if  $F = 1$ , we must have  $F \neq 1$ . For a wave of elevation,  $\eta(x) \geq 0$  for all  $x$  but  $\eta \not\equiv 0$ , so (3.7) implies that the coefficient  $1/F^2 - 1$  is strictly negative, i.e. that  $F > 1$ . Similarly, for a wave of depression,  $\eta(x) \leq 0$  for all  $x$  but  $\eta \not\equiv 0$ , so (3.7) implies that  $1/F^2 - 1$  is strictly positive, i.e. that  $F < 1$ .  $\square$



## 4 Upper bound

In this section we will make use of the vorticity function  $\gamma(-p)$ , Bernoulli constant  $\lambda = (c - U(0))^2$ , and antiderivative  $\Gamma(p)$  of  $\gamma$  defined in Section 2. We begin by giving a formula in our notation for an invariant called the flow force.

**Lemma 4.1.** *For any solitary wave with  $\sup u < c$  and the regularity (1.2) and for any  $x$ , the flow force*

$$S := \int_{-d}^{\eta(x)} (P + (u - c)^2)(x, y) dy = -2 \int_{-m}^0 \gamma H dp + \lambda d + \frac{gd^2}{2}. \quad (4.1)$$

*Proof.* That  $S$  is independent of  $x$  is well-known, and can be proved, for instance, by integrating the identity  $(P + (u - c)^2)_x + ((u - c)v)_y = 0$  over a region of the form  $\{(x, y) : a < x < b, -d < y < \eta(x)\}$  using the divergence theorem and then applying the boundary conditions. To obtain the formula (4.1), we use Bernoulli's law (2.10) to rewrite

$$P + (u - c)^2 = \frac{(u - c)^2 - v^2}{2} - gy + \frac{\lambda}{2} + \Gamma = \frac{1 - h_q^2}{2h_p^2} - g(h - d) + \frac{\lambda}{2} + \Gamma.$$

The asymptotic condition (2.5d) gives

$$\frac{1 - h_q^2}{2h_p^2} \rightarrow \frac{1}{2H_p^2} = \frac{\lambda}{2} + \Gamma \text{ as } q \rightarrow \pm\infty,$$

uniformly in  $p$ , and hence, since  $S$  is independent of  $x$ ,

$$\begin{aligned} S &= \lim_{q \rightarrow \pm\infty} \int_{-m}^0 \left( \frac{1 - h_q^2}{2h_p^2} - g(h - d) + \frac{\lambda}{2} + \Gamma \right) h_p dp \\ &= \int_{-m}^0 (\lambda + 2\Gamma - g(H - d)) H_p dp, \\ &= -2 \int_{-m}^0 \gamma H dp + \lambda d + \frac{gd^2}{2}, \end{aligned} \quad (4.2)$$

where in the last step we integrated by parts using  $H(-m) = \Gamma(0) = 0$  and  $\Gamma_p = \gamma$ .  $\square$

The following integral identity is (1.6) but with an extra term coming from the vorticity. Unlike Lemmas 3.1 and 4.3, it is proved mostly in the original physical variables.

**Lemma 4.2.** *Any solitary wave with  $\sup u < c$  and the regularity (1.2) satisfies*

$$(\lambda - gd) \int_{-M}^M \eta dx - \frac{3g}{2} \int_{-M}^M \eta^2 dx - 2 \int_{-M}^M \int_{-m}^0 \gamma(h - H) dp dq \rightarrow 0 \quad \text{as } M \rightarrow \infty. \quad (4.3)$$

*Proof.* Consider the fluid region

$$D = \{(x, y) \in \mathbb{R}^2 : -M < x < M, -d < y < \eta(x)\},$$

and the two  $W^{1,r}(D)$  vector fields

$$\begin{aligned} A &= (A^1, A^2) = (P + (u - c)^2, (u - c)v), \\ B &= (B^1, B^2) = ((u - c)v, P + v^2 + gy). \end{aligned}$$

By the incompressible Euler equations (2.1a)–(2.1c),  $A$  and  $B$  are both divergence free. Thus

$$\begin{aligned}\operatorname{div}(xA + (y + d)B) &= A^1 + B^2 \\ &= 2P + (u - c)^2 + v^2 + gy \\ &= \lambda + 2\Gamma(-\psi) - gy\end{aligned}\tag{4.4}$$

in  $L^r(D)$ , where in the last step we have used Bernoulli's law (2.10). Integrating (4.4) over  $D$ , the divergence theorem gives

$$\int_{\partial D} (xA + (y + d)B) \cdot n \, ds = \iint_D (\lambda - gy + 2\Gamma(-\psi)) \, dy \, dx,\tag{4.5}$$

where  $n$  is an outward pointing normal. We will obtain (4.3) by simplifying both sides of (4.5) and using Lemma 4.1.

First consider the left hand side of (4.5). On the free surface, the two boundary conditions  $v = \eta_x(u - c)$  and  $P = 0$  give

$$(xA + (y + d)B) \cdot n = (xA + (y + d)B) \cdot \frac{(v, c - u)}{\sqrt{(u - c)^2 + v^2}} = \frac{gy(y + d)(c - u)}{\sqrt{(u - c)^2 + v^2}},$$

while on the bottom  $y = -d$  the boundary condition  $v = 0$  implies  $A^2 = (y + d)B^2 = 0$ . Thus we see

$$\begin{aligned}\int_{\partial D} (xA + (y + d)B) \cdot n \, ds &= x \int_{-d}^{\eta(x)} (P + (u - c)^2) \, dy \Big|_{x=-M}^{x=M} \\ &\quad + \int_{-d}^{\eta(x)} (y + d)(u - c)v \, dy \Big|_{x=-M}^{x=M} + \int_{-M}^M g(\eta + d)\eta \, dx.\end{aligned}\tag{4.6}$$

By Lemma 4.1, we have

$$\int_{-d}^{\eta(x)} (P + (u - c)^2) \, dy = -2 \int_{-m}^0 \gamma(-p)H \, dp + \lambda d + \frac{gd^2}{2}$$

for all  $x$ , and since  $v \rightarrow 0$  uniformly in  $y$  as  $x \rightarrow \pm\infty$ , the second term on the right hand side of (4.6) vanishes as  $M \rightarrow \infty$ . Thus (4.6) implies

$$\begin{aligned}\int_{\partial D} (xA + (y + d)B) \cdot n \, ds &- g \int_{-M}^M \eta^2 \, dx \\ &- gd \int_{-M}^M \eta \, dx - 2M \left( \lambda d + \frac{gd^2}{2} - 2 \int_{-m}^0 \gamma H \, dp \right) \rightarrow 0\end{aligned}\tag{4.7}$$

as  $M \rightarrow \infty$ .

Now we turn to the right hand side of (4.5). Changing variables and integrating by parts,

$$\int_{-d}^{\eta(x)} \Gamma(-\psi) \, dy = \int_{-m}^0 \Gamma(p)h_p \, dp = - \int_{-m}^0 \gamma(-p)h \, dp,$$

where we have used that  $\Gamma$  vanishes on  $p = 0$  while  $h$  vanishes on  $p = -m$ . Thus

$$\begin{aligned}\iint_D (\lambda - gy + 2\Gamma(-\psi)) \, dy \, dx &= \lambda \int_{-M}^M \eta \, dx - \frac{g}{2} \int_{-M}^M \eta^2 \, dx - 2 \int_{-M}^M \int_{-m}^0 \gamma h \, dp \, dx \\ &\quad + 2M \left( \lambda d + \frac{gd^2}{2} \right)\end{aligned}\tag{4.8}$$

Substituting (4.7) and (4.8) into (4.5), most of the terms drop out and we are left with (4.3) as desired.  $\square$

**Lemma 4.3.** *Any solitary wave with  $u < c$  and the regularity (1.2) satisfies*

$$\frac{3g}{2} \int_{-M}^M \eta^2 dx - \int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2H_p^2 h_p^2} dp dq \rightarrow 0 \quad \text{as } M \rightarrow \infty. \quad (4.9)$$

*Proof.* We argue as in the proof of Lemma 3.1, but with the function  $\Phi$  replaced by  $H$ , and then appeal to Lemma 4.2. Multiplying (2.5a) by  $H$  and integrating by parts, we have, for any  $M > 0$ ,

$$\begin{aligned} 0 &= \int_{-M}^M \int_{-m}^0 \left[ \left( -\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right)_p H + \left( \frac{h_q}{h_p} \right)_q H \right] dp dq \\ &= \int_{-M}^M \int_{-m}^0 \left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p dp dq + gd \int_{-M}^M \eta dq + \int_{-m}^0 \frac{h_q}{h_p} H dp \Big|_{x=-M}^{x=M}, \end{aligned} \quad (4.10)$$

where we have used the boundary condition (2.5b) as well as  $h(q, 0) - H(0) = \eta(q)$  and  $H(0) = d$ . As in the proof of Lemma 3.1, the asymptotic conditions (2.5d) imply that the last term in (4.10) vanishes as  $M \rightarrow \infty$ . The integrand in the first term of (4.10) can be rewritten as

$$\left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p = -\frac{h_p - H_p}{H_p^2} + \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2 H_p^2}, \quad (4.11)$$

as can be seen by dividing (3.5) by  $H_p^2$ . Since

$$\left( \frac{1}{H_p^2} \right)_p = (\lambda + 2\Gamma)_p = 2\gamma, \quad \frac{1}{H_p^2(0)} = \lambda,$$

we can integrate the first term of (4.11) by parts to get

$$\begin{aligned} - \int_{-m}^0 \frac{h_p - H_p}{H_p^2}(q, p) dp &= 2 \int_{-m}^0 \left( \frac{1}{H_p^2} \right)_p (h(q, p) - H(p)) dp - \lambda(h(q, 0) - H(0)) \\ &= 2 \int_{-m}^0 \gamma(h(q, p) - H(p)) dp - \lambda\eta(q). \end{aligned}$$

Putting everything together, we see that (4.10) implies

$$\begin{aligned} &\int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2 H_p^2} dp dq + 2 \int_{-M}^M \int_{-m}^0 \gamma(h - H) dp dq \\ &\quad + (gd - \lambda) \int_{-M}^M \eta dq \longrightarrow 0 \quad \text{as } M \rightarrow \infty \end{aligned} \quad (4.12)$$

Adding (4.12) with the conclusion (4.3) of Lemma 4.2, most of the terms cancel and we are left with (4.9) as desired.  $\square$

Using Lemmas 3.1 and 4.3, we can now prove Theorem 1.2.

*Proof of Theorem 1.2.* The key observation is that the integrands of the double integrals in (3.1) and (4.9) differ only by a factor of  $H_p^2$ . Plugging the crude estimate

$$\begin{aligned} &\int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2} dp dq \\ &\leq \max H_p^2 \int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2 H_p^2} dp dq \end{aligned}$$

into the result (3.1) of Lemma 3.1, we obtain

$$\limsup_{M \rightarrow \infty} \left\{ \left(1 - \frac{1}{F^2}\right) \int_{-M}^M \eta \, dx - \max H_p^2 \int_{-M}^M \int_{-m}^0 \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2 H_p^2} \, dp \, dq \right\} \leq 0.$$

Subtracting the conclusion (4.9) of Lemma 4.3 multiplied by  $\max H_p^2$ , we then get

$$\limsup_{M \rightarrow \infty} \left\{ \left(1 - \frac{1}{F^2}\right) \int_{-M}^M \eta \, dx - \frac{3g \max H_p^2}{2} \int_{-M}^M \eta^2 \, dq \right\} \leq 0. \quad (4.13)$$

But from Bernoulli's law (2.10) evaluated on the free surface we have  $g \max \eta < \lambda/2$ , so that (4.13) implies

$$\limsup_{M \rightarrow \infty} \left(1 - \frac{1}{F^2} - \frac{3\lambda}{4} \max H_p^2\right) \int_{-M}^M \eta \, dx < 0,$$

and hence

$$1 - \frac{1}{F^2} - \frac{3\lambda}{4} \max H_p^2 < 0. \quad (4.14)$$

By (2.7) and (2.8),  $\lambda = (U - c)^2(0)$  and  $H_p^2 = (U - c)^{-2}$ , so

$$\lambda \max_p H_p^2 = \max_y \frac{(c - U(0))^2}{(c - U(y))^2} = \Lambda^2,$$

and (4.14) becomes

$$1 - \frac{1}{F^2} - \frac{3\Lambda^2}{4} < 0,$$

which, assuming  $\Lambda < 2/\sqrt{3}$ , is equivalent to the desired upper bound on  $F$  in (1.4).  $\square$

## 5 Amplitude, elevation, symmetry, monotonicity, and decay

In this section we will give several corollaries of Theorems 1.1–1.2 and their proofs. Some of these will require stronger regularity and decay assumptions than (1.2) and (2.1g), namely

$$h \in C^{2+\alpha}(\overline{\Omega}), \quad H \in C^{2+\alpha}[-d, 0], \quad (5.1)$$

$$h - H, \, D(h - H), \, D^2(h - H) \rightarrow 0 \text{ as } q \pm \infty, \quad (5.2)$$

where the limit in (5.2) is uniform in  $p$ .

**Corollary 5.1** (Bound on the amplitude). *In the setting of Theorem 1.2, the maximum amplitude  $\max_x \eta(x)$  satisfies the following upper bound*

$$\frac{\max \eta}{d} < \frac{(c - U(0))^2}{2gd} < \frac{1}{2} \frac{\Lambda^2}{1 - \frac{3}{4}\Lambda^2}. \quad (5.3)$$

*Proof.* The first inequality in (5.3) is just Bernoulli's law (2.10) evaluated on the free surface. Next, we note that the definitions (1.1), (1.3), and (2.7) of  $F$  and  $\Lambda$  immediately imply the simple inequality  $(c - U(0))^2 \leq gd\Lambda^2 F^2$ . The second inequality in (5.3) then follows from the upper bound on  $F$  in Theorem 1.2.  $\square$

When  $U(y) \leq U(0)$  for  $-d \leq y \leq 0$ , such as for instance when  $\omega \leq 0$  so that  $U_y \geq 0$ , we have  $\Lambda = 1$  and hence that the second inequality in (5.3) is the simple bound  $\max \eta \leq 2d$ . See Section 7 for the case of constant vorticity.

**Corollary 5.2** (Elevation). *A solitary wave with  $\sup u < c$  and the regularity (5.1) and decay (5.2) is a wave of elevation if and only if  $F > 1$ , and in this case it is a strict wave of elevation in that  $\eta(x) > 0$  for all  $x \in \mathbb{R}$ .*

*Proof.* Thanks to Theorem 1.1, this is an immediate consequence of Proposition 2.1 in [Whe13], which states that all waves with  $F \geq 1$  and the above regularity and decay are strict waves of elevation.  $\square$

While there are no symmetric and monotone irrotational solitary waves of depression [KP74], it is an open question if the same is true with vorticity. More generally, it is unknown if there are any solitary waves with subcritical Froude number  $F < 1$ . By Theorem 1.1, no such wave could be a wave of elevation, and in fact (3.1) implies something stronger.

**Corollary 5.3** (Negative mass for subcritical waves). *Any nontrivial solitary wave with  $\sup u < c$ , the regularity (1.2), and subcritical Froude number  $F < 1$  must have*

$$\limsup_{M \rightarrow \infty} \int_{-M}^M \eta \, dx < 0.$$

*Proof.* This follows immediately from (3.7).  $\square$

The following corollary is interesting only in the method of proof; the conclusion of Corollary 5.5, which follows from [Hur08b], is much stronger.

**Corollary 5.4** (Finite mass). *For any solitary wave of elevation with  $\sup u < c$  and the regularity (1.2), all of the definite integrals appearing in Lemmas 3.1, 4.2, and 4.3, have a (finite) limit as  $M \rightarrow \infty$ . In particular, the limits in (3.1), (4.3), and (4.9) become equalities when  $M$  is replaced by  $+\infty$ .*

*Proof.* We argue as in Section 3 of [McL84]. Assume for contradiction that

$$\int_{-M}^M \eta \, dx \rightarrow \infty \text{ as } M \rightarrow \infty.$$

Since  $\eta \rightarrow 0$  as  $x \rightarrow \pm\infty$ , we have

$$\frac{\int_{-M}^M \eta^2 \, dx}{\int_{-M}^M \eta \, dx} \rightarrow 0 \text{ as } M \rightarrow \infty.$$

But then, since  $F > 1$  by Theorem 1.1, the left hand side of (4.13) tends to  $+\infty$  as  $M \rightarrow \infty$ , a contradiction. Thus  $\int_{-\infty}^{\infty} \eta \, dx$  and hence  $\int_{-\infty}^{\infty} \eta^2 \, dx$  are both finite. The statement then follows by combining this result with the limits in (3.1), (4.3), and (4.9).  $\square$

In order to state the final corollary in this section, we introduce the Sturm-Liouville problem

$$\begin{aligned} \left( \frac{\varphi_p}{H_p^3} \right)_p + \mu \frac{\varphi_p}{H_p} &= 0 \text{ for } -m < p < 0, \\ \frac{\varphi_p(0)}{H_p^3} - g\varphi(0) &= 0, \quad \varphi(-m) = 0, \end{aligned} \tag{5.4}$$

which appears when studying the linearization of (2.5) around  $h = H$ . We define  $\mu_1, \mu_2$  to be the smallest and second smallest eigenvalues of (5.4), and  $\varphi_1$  to be the eigenfunction corresponding to  $\mu_1$ . When  $F > 1$ ,  $\mu_1$  is positive, and we can take  $\varphi_1$  to be positive for  $-m < p \leq 0$  [Hur08b]. We note that (5.4) is equivalent to the system

$$\begin{aligned} (U - c)(\tilde{\varphi}_{yy} + \mu\tilde{\varphi}) - U_{yy}\tilde{\varphi} &= 0 \text{ for } -d < y < 0, \\ (U - c)^2\tilde{\varphi}_y(0) - (g + (U - c)U_y)\tilde{\varphi}(0) &= 0, \quad \tilde{\varphi}(-d) = 0, \end{aligned}$$

for  $\tilde{\varphi}(y) := (c - U(y))\varphi(p)$  in the original physical variables; see Lemma 2.3 in [HL08].

**Corollary 5.5** (Symmetry, monotonicity, and decay). *Any solitary wave of elevation with  $\sup u < c$  and the regularity (5.1) and decay (5.2) has the following properties.*

- (a) *(Symmetry and monotonicity) The wave is symmetric and monotone in that, after shifting the definition of the horizontal variable  $q$ , the height function  $h$  is even in  $q$  and has  $h_q < 0$  for  $q > 0$  and  $-m < p \leq 0$ . In particular, after this shift  $\eta$  is an even function of  $x$  with  $\eta_x < 0$  for  $x > 0$ .*
- (b) *(Decay and asymptotics) The difference  $w := h - H$  decays exponentially as  $q \rightarrow \pm\infty$ , and satisfies the asymptotic estimate*

$$|D^k(w(q, p) - r\varphi_1(p)e^{-\sqrt{\mu_1}|q|})| \leq Ce^{-s_1|q|} \quad \text{for } k \leq 1, \quad |q| > 1,$$

for some constants  $C, r > 0$  depending on  $w$ , where  $\mu_1 > 0$  and  $\varphi_1$  are defined above in terms of the Sturm-Liouville problem (5.4) and the exponent  $s_1$  appearing on the right hand side satisfies  $\sqrt{\mu_1} < s_1 < \min(2\sqrt{\mu_1}, \sqrt{\mu_2})$ .

*Proof.* Since the waves considered in this corollary have  $F > 1$  by Theorem 1.1, part 5.5 follows immediately from Theorem 3.1 in [Hur08b], while part 5.5 follows from Proposition 4.6 in the same paper.  $\square$

## 6 Existence of large-amplitude waves

In this section we observe the implications of Theorems 1.1 and 1.2 for the existence theory of large-amplitude solitary waves with vorticity developed in [Whe13]. This theory involves a one parameter family of shear flows

$$U(y) = c - FU^*(y), \tag{6.1}$$

where  $U^*$  is an arbitrary but fixed strictly positive function satisfying the normalization condition

$$g \int_{-d}^0 \frac{dy}{U^*(y)^2} = 1. \tag{6.2}$$

The normalization (6.2) ensures that the parameter  $F$  in (6.1) is indeed the Froude number  $F$  defined in (1.1). In the corollary below we call a solitary wave *symmetric* if  $u$  and  $\eta$  are even in  $x$ , and  $v$  is odd in  $x$ , and *monotone* if in addition  $\eta(x)$  is strictly decreasing for  $x > 0$ .

**Corollary 6.1** (Existence of large-amplitude waves). *Fix  $g, c, d > 0$ , a Hölder parameter  $0 < \alpha \leq 1/2$ , and a strictly positive function  $U^* \in C^{2+\alpha}[-d, 0]$  satisfying the normalization condition (6.2). Then there exists a connected set  $\mathcal{C}$  of solitary waves*

$$(u, v, \eta, F) \in C^{1+\alpha} \times C^{1+\alpha} \times C^{2+\alpha}(\mathbb{R}) \times (1, \infty),$$

where  $F$  determines the asymptotic shear flow  $U$  via (6.1), with the following properties. Each wave in  $\mathcal{C}$  is a symmetric and monotone wave of elevation with  $\sup u < c$  and  $F > 1$ . Moreover, at least one of following two conditions holds:

- (i) (Stagnation) *There is a sequence of flows  $(u_n, v_n, \eta_n, F_n) \in \mathcal{C}$  and sequence of points  $(x_n, y_n)$  such that  $u_n(x_n, y_n) \nearrow c$ ; or*
- (ii) (Large Froude number) *There exists a sequence of flows  $(u_n, v_n, \eta_n, F_n) \in \mathcal{C}$  with  $F_n \nearrow \infty$ .*

*Proof.* By Theorem 1.1 in [Whe13], it is enough to show that no solitary wave  $(u, v, \eta, F)$  in the closure of  $\mathcal{C}$  can have the critical Froude number  $F = 1$ . Let  $(u, v, \eta, F)$  be a wave in the closure of  $\mathcal{C}$ . By Proposition 2.4 in [Whe13],  $\sup u < c$ , so Theorem 1.1 implies that  $F \neq 1$ .  $\square$

We note that, even when (i) occurs in Corollary 6.1, the shear flow  $U$  is bounded away from  $c$  uniformly along the continuum  $\mathcal{C}$ . Indeed, every wave in  $\mathcal{C}$  has  $F > 1$  and hence

$$\min(c - U) = F \min U^* > \min U^* > 0.$$

In many cases, we can apply Theorem 1.2 to further simplify Corollary 6.1.

**Corollary 6.2.** *In the setting of Corollary 6.1, suppose that the fixed profile  $U^*$  satisfies*

$$\Lambda^* := \max_y \frac{U^*(0)}{U^*(y)} < \frac{2}{\sqrt{3}}.$$

*Then condition (ii) cannot occur, so that (i) must hold.*

*Proof.* Thanks to (6.1), any wave in  $\mathcal{C}$  satisfies

$$\Lambda = \max_y \frac{c - U(0)}{c - U(y)} = \max_y \frac{U^*(0)}{U^*(y)} = \Lambda^*.$$

Thus by Theorem 1.2 all waves in  $\mathcal{C}$  have  $F < (1 - \frac{3}{4}(\Lambda^*)^2)^{-1/2} < \infty$ .  $\square$

The conclusion of Corollary 6.2, that there exists a sequence of solutions along the continuum with  $\sup u_n$  approaching  $c$ , is the same conclusion that was proved for periodic waves by Constantin and Strauss in [CS04]. It remains an open question if the same is true for solitary waves with  $\Lambda^* \geq 2/\sqrt{3}$ , though it seems doubtful that the restriction  $\Lambda^* < 2/\sqrt{3}$  is sharp.

## 7 The case of constant vorticity

In this section we specialize the above results to asymptotic shear flows  $U$  which are linear in  $y$ , or equivalently to waves whose vorticity

$$\omega(x, y) = \gamma(p) \equiv -U_y$$



is constant. In this case more explicit formulas are available; to make these formulas appear simpler, we define the dimensionless constants

$$\lambda^* = \frac{\lambda}{gd} = \frac{(c - U(0))^2}{gd} > 0, \quad \gamma^* = \frac{\gamma d}{\sqrt{\lambda}} = \frac{-U_y d}{(c - U(0))}.$$

In terms of  $\lambda^*$  and  $\gamma^*$ , the asymptotic shear flow  $U$  is given by

$$c - U(y) = \sqrt{\lambda} + \gamma y = \sqrt{gd\lambda^*} \left(1 + \gamma^* \frac{y}{d}\right). \quad (7.1)$$

Plugging  $y = -d$  into (7.1), we see that  $\sup U < c$  implies  $\gamma^* < 1$ . Substituting (7.1) into the definitions (1.1) of  $F$  and (1.3) of  $\Lambda$ , we easily check that

$$F^2 = \lambda^*(1 - \gamma^*), \quad \Lambda = \frac{1}{1 - \max(\gamma^*, 0)}. \quad (7.2)$$

Using (7.2) in Theorems 1.1 and 1.2, we can then easily prove the following.

**Corollary 7.1** (Constant vorticity). *Consider a solitary wave of elevation with constant vorticity  $\gamma$ , regularity (1.2), and  $\sup u < c$  so that in particular  $\gamma^* < 1$ . Then we have the following bounds on  $\lambda^*$  and  $\gamma^*$ ,*

$$\lambda^*(1 - \gamma^*) > 1, \quad (7.3)$$

$$\lambda^*(1 - \gamma^*) < 4 \quad \text{for } \gamma^* \leq 0, \quad (7.4)$$

$$\lambda^* \frac{1 - 8\gamma^* + 4(\gamma^*)^2}{1 - \gamma^*} < 4 \quad \text{for } 0 < \gamma^* < 1 - \sqrt{3}/2, \quad (7.5)$$

and hence the bounds

$$\frac{\max \eta}{d} < \frac{2}{1 - \gamma^*} \quad \text{for } \gamma^* \leq 0, \quad \frac{\max \eta}{d} < \frac{2(1 - \gamma^*)}{1 - 8\gamma^* + 4(\gamma^*)^2} \quad \text{for } 0 < \gamma^* < 1 - \sqrt{3}/2. \quad (7.6)$$

on the amplitude.

*Proof.* The first inequality (7.3) is  $F > 1$  from Theorem 1.1, while (7.4) and (7.5) are (1.4) from Theorem 1.2. Since Bernoulli's law (2.10) implies  $\max \eta < d\lambda^*/2$ , the bounds (7.6) on the amplitude follow immediately from (7.4) and (7.5).  $\square$

From (7.4) and (7.5) we see that, for any fixed  $\gamma^* < 1 - \sqrt{3}/2 \approx 0.134$ , waves of elevation with  $\sup u < c$  have  $\lambda^*$  bounded above by a constant depending only on  $\gamma^*$ . It is interesting to compare this to the numerical results in [VB94], which suggest that for any fixed  $\gamma^* > \gamma_{\text{cr}}^* \approx 0.33$  there exist overhanging waves with  $\lambda^*$  arbitrarily large. Note however that overhanging waves necessarily violate our assumption  $\sup u < c$ .

## 8 Surface tension

In this section we prove that Theorem 1.1 continues to hold in the presence of surface tension. For waves with surface tension, the dynamic boundary condition (2.1f) is replaced by

$$P + \sigma \left( \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right)_x = 0 \quad \text{on } y = \eta(x), \quad (8.1)$$

where the constant  $\sigma$  is the coefficient of surface tension. In the following we permit  $\sigma$  to be positive or negative. The corresponding boundary condition (2.5b) in the height equation becomes

$$\frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} + g(h-H) - \sigma \left( \frac{h_q}{(1+h_q^2)^{1/2}} \right)_q = 0 \quad \text{on } p=0. \quad (8.2)$$

For irrotational solitary waves with surface tension, Amick and Kirchgässner proved the analogue

$$F^2 = 1 + \frac{3}{2d} \frac{\int \eta^2 dx}{\int \eta dx} + \frac{\sigma}{gd} \frac{\int (\sqrt{1+\eta_x^2} - 1) dx}{\int \eta dx} \quad (8.3)$$

of the integral identity (1.6) in [AK89]. For irrotational waves of depression, such as those constructed in [AK89], (8.3) immediately implies that the Froude number  $F < 1$ . As a consequence of Theorem 8.1 below, the same bound  $F < 1$  holds for waves with vorticity.

We will work with solutions which have the regularity

$$\eta = h(\cdot, 0) \in W_{\text{loc}}^{2,r}(\mathbb{R}), \quad h \in W_{\text{loc}}^{2,r}(\overline{\Omega}) \subset C_{\text{loc}}^{1+\alpha}(\overline{\Omega}), \quad H \in W^{2,r}[-m, 0], \quad (8.4)$$

where, as in Section 1.1,  $0 < \alpha < 1$  and  $r = 2/(1-\alpha)$ ; see [MM14] for the equivalence of various formulations for periodic waves with surface tension.

**Theorem 8.1.** *Theorem 1.1 holds for waves with surface tension, provided we replace the regularity (1.2) with (8.4).*

*Proof.* We argue exactly as in the proof of Theorem 1.1, with Lemma 3.1 replaced by Lemma 8.2 below.  $\square$

**Lemma 8.2.** *Lemma 3.1 holds for waves with surface tension, provided we replace the regularity (1.2) with (8.4).*

*Proof.* We will follow the proof of Lemma 3.1 and notice that the term involving surface tension drops out of the calculation entirely. Multiplying (2.5a) by  $\Phi(p) = \int_{-m}^p H_p^3(s) ds$  and integrating by parts, we obtain (3.3) as before,

$$\begin{aligned} 0 &= \int_{-M}^M \int_{-m}^0 \left( \frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \right) \Phi_p dp dq + \frac{1}{gF^2} \int_{-M}^M \left( -\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right) (q, 0) dq \\ &\quad + \int_{-m}^0 \frac{h_q}{h_p} \Phi dp \Big|_{x=-M}^{x=M}. \end{aligned} \quad (8.5)$$

We claim that (8.5) implies (3.4). Indeed, the first and last terms in (8.5) can be treated as in the proof of Lemma 3.1, so (3.4) follows from the following computation involving the middle term,

$$\begin{aligned} &\frac{1}{gF^2} \int_{-M}^M \left( -\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \right) (q, 0) dq - \frac{1}{F^2} \int_{-M}^M (h(q, 0) - H(0)) dq \\ &= -\frac{\sigma}{gF^2} \int_{-M}^M \left( \frac{h_q}{(1+h_q^2)^{1/2}} \right)_q (q, 0) dq \\ &= -\frac{\sigma}{gF^2} \frac{h_q}{(1+h_q^2)^{1/2}} \Big|_{(-M, 0)}^{(M, 0)} \rightarrow 0 \text{ as } M \rightarrow \infty, \end{aligned}$$

where we have used the boundary condition (8.2) and the asymptotic conditions (2.5d). With (3.4) established, we can then complete the argument exactly as in the proof of Lemma 3.1.  $\square$

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